A $O(\mathrm{n}^8) \times O(\mathrm{n}^7)$ Linear Programming Model of the Traveling Salesman Problem

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Abstract: In this paper, we propose a new linear programming (LP) formulation of the Traveling Salesman Problem (TSP). The proposed model has $O(n^8)$ variables and $O(n^7)$ constraints, where n is the number of cities. Our numerical experimentation shows that computational times for the proposed linear program are several orders of magnitude smaller than those for the existing model [3].

Keywords: Traveling Salesman Problem; Linear Programming; Combinatorial Optimization; Computational Complexity.

1 Introduction

The Traveling Salesman Problem (TSP) is the problem of finding a least-cost sequence in which to visit a set of cities, starting and ending at the same city, and in such a way that each city is visited exactly once. Interest in this problem cuts across several fields of study in Business, Economics, Engineering, and Computer Science, due partly to the wide applicability of the problem in practice (see [7]), and partly to the central place the problem occupies with respect to theoretical developments in the areas of Combinatorial Optimization (see [8]) and Computational Complexity (see [5] or [10], among others). The seminal Operations Research work on the problem is that of ([4]). A linear programming (LP) formulation was proposed in [3]. That formulation has $O(n^9)$ variables and $O(n^8)$ constraints (where n is the number of cities).

In this paper we propose a new linear programming formulation of the Traveling Salesman Problem. The modeling approach used in the paper is similar to that of [3]. However, the proposed model is an order of size smaller, having $O(n^8)$ variables and $O(n^7)$ constraints. A small numerical experimentation we conducted shows that the computational times for the linear program proposed in this paper are several orders of magnitude smaller than those for the model in [3].

The plan of the paper is as follows. The proposed linear programming formulation is developed in section 2. The numerical experimentation is discussed in section 3. Conclusions are discussed in section 4.

Notation 1 *The following notation will be used throughout the rest of this paper:*

1. The number of cities is denoted n;

2. The set of cities is denoted $N := \{1, 2, \ldots, n\}$;

3. The set of all TSP tours is denoted Δ;

4. The cost of travel from city i to city j ($j \neq i$) is denoted x_{ij};

5. The set of real numbers is denoted by $\mathcal{R}$;
6. For two column vectors **a** and **b**, $\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = (\mathbf{a}^T, \mathbf{b}^T)^T$ will be written as "($\mathbf{a}$, $\mathbf{b}$)" (where $(\cdot)^T$denotes the transpose of $(\cdot)$), except for where that causes ambiguity;
7. The i^{th} component of a column vector **a** is denoted $\mathbf{a}_i$;
8. The notation "$\mathbf{0}$" denotes a column vector of comfortable size that has every entry equal to 0;
9. The notation "$\mathbf{1}$" denotes a column vector of comfortable size that has every entry equal to 1;
10. The convex hull of $(\cdot)$ is denoted $Conv(\cdot)$;
11. The set of extreme points of $(\cdot)$ is denoted $Ext(\cdot)$.

2 Development of the Formulation

Assumption 2 *We assume without loss of generality (w.l.o.g.) that:*

1. The number of cities is greater than 5 (i.e., $n \geq 6$);
2. *City* 1 is the starting and ending point of travel;
3. All vectors of variables are column vectors.

Notation 3

1. $M := N \backslash \{1\} = \{2, ..., n\}$
2. The set of "times-of-travel" to the cities in M is denoted $T := \{1, ..., n-1\}$; (i.e., if the time-of-travel to *city* $i \in M$ is r ($r \in T$), then i is the r^{th} city to be visited after *city* 1);
3. The 0/1 binary that indicates whether *city* i is visited at *time(-of-travel)* r is denoted w_{ir}; ($w_{ir} = 1$ iff *city* i is visited at *time* r);

Theorem 4 *There exists a one-to-one correspondence between the TSP tours and the perfect matchings of M and T;That is, there exists a one-to-one correspondence between TSP tours and points of*

$$W_I := \left\{ w \in \{0,1\}^{(n-1)^2} : \sum_{i \in M} w_{ir} = 1, \ r \in T; \ \ \sum_{r \in T} w_{ir} = 1, \ i \in M \right\}. \tag{1}$$

Proof. Trivial. ■

The basic idea of our approach in this paper is to model the TSP as an optimization problem over a reformulation of the linear programming relaxation,

$$W_L := \left\{ w \in [0,1]^{(n-1)^2} : \sum_{i \in M} w_{ir} = 1, \ r \in T; \ \ \sum_{r \in T} w_{ir} = 1, \ i \in M \right\}, \tag{2}$$

of W_I. Hence, we do not deal with the TSP Polytope per se (see [6, pp. 256-261]). Hence, results pertaining to descriptions of the TSP Polytope specifically (see [6], [9], and [11], in particular) are not applicable to the developments in this paper.

Theorem 5 *The following hold true:*

1. $Ext(W_I) = W_I$;
2. $Ext(W_L) = Ext(W_I)$;
3. $Conv(W_L) = Conv(W_I)$.

Proof. See [1] or [8]. ■

Definition 6 *We refer to* $Conv(W_L) = Conv(W_I)$ *as the "Assignment Polytope."*

As in [3], in order to reformulate W_I and W_L, we use the framework of the multipartite digraph G = (V, A) illustrated in *Figure 1.* Each node in this graph is defined by a (city, time-of-travel) pair. Arcs are labeled with triplets $(a, b, c) \in (M, T\backslash\{n-1\}, M)$, and represent the possible choices at consecutive times-of-travel. Specifically, the arc linking nodes (i, r) and $(j, r+1)$ of the graph is labeled (i, r, j), and represents the choice to visit cities i and j at travel times r and $r+1$, respectively.

Definition 7

1. The set of all the nodes of *Graph G* that have a given city index in common is referred to as a *level* of the graph. The i^{th} level $(i \in M)$ is denoted $L_i(G) := \{(u, v) \in V \mid u = i\}$;
2. The set of all the nodes of *Graph G* that have a given time-of-travel index in common is referred to as a *stage* of the graph. The j^{th} stage $(j \in T)$ is denoted $S_j(G) := \{(u, v) \in V \mid v = j\}$;
3. A path in *Graph G* that simultaneously spans the *levels* and the *stages* of *Graph G* is referred to as a *"city and stage spanning (c.a.s.s.) path"* of the graph;
4. The set of all the *c.a.s.s. paths* of *Graph G* is denoted Ω. That is,
$\Omega := \{((i_1, 1, i_2), (i_2, 2, i_3), ..., (i_{n-2}, n-2, i_{n-1})) \in A^{n-2} : i_p \neq i_q \ \forall \ (p, q) \in (T, T\backslash\{p\})\}$.

Remark 8 *The association we make in our modeling between* Graph G *and the TSP is to interpret a positive flow into/out of any node of the graph to mean that the corresponding city and time-of-travel pair have been assigned to each other.*

Remark 9

1. There exists a one-to-one correspondence between the perfect matchings of M and T and the *c.a.s.s. paths* of *Graph G*;
2. There exists a one-to-one correspondence between the *c.a.s.s. paths* of *Graph G* and the points of $Ext(W_L) = Ext(W_I) = W_I$;
3. There exists a one-to-one correspondence between the *c.a.s.s. paths* of *Graph G* and TSP tours.

Figure 1: Illustration of Graph G

2.1 Reformulation of the Assignment Polytope

Notation 10 *:*

1. The set of *stages* of *Graph G* from which arcs of the graph originate is denoted R; i.e., $R := T\backslash\{n-1\}$;

2. $\forall\ (i,j,u,v,k,t) \in F^6$, $\forall\ (p,s) \in R^2 : 1 < p < s$, $z_{i,1,jupvkst}$ denotes a non-negative variable that represents the amount of flow in *Graph G* that propagates from arc $(i,1,j)$ onto arc (k,s,t), via arc (u,p,v);

3. $\forall\ (i,j,k,t) \in F^4$, $\forall\ (r,s) \in R^2 : 1 \leq r < s$, y_{irjkst} denotes a non-negative variable that represents the total amount of flow in *Graph G* that propagates from *arc* (i,r,j) onto *arc* (k,s,t).

2.1.1 Integer Programming Reformulation

The constraints of our Integer Programming reformulation of the Assignment Polytope are as follows:

$$\sum_{i\in M}\sum_{j\in M}\sum_{v\in M}\sum_{t\in M} z_{i,1,jj,2,vv,3,t} = 1 \tag{3}$$

$$\sum_{v\in M} z_{i,1,jkstvpu} - \sum_{v\in M} z_{i,1,jkstu,p+1,v} = 0;$$
$$i,j,k,t,u \in M; \quad p,s \in R : 1 < s < p < n-2 \tag{4}$$

$$\sum_{v\in M} z_{i,1,jvpukst} - \sum_{v\in M} z_{i,1,ju,p+1,vkst} = 0;$$
$$i,j,k,t,u \in M; \quad p,s \in R : 1 < p < s-1, \;\; s > 3 \tag{5}$$

$$y_{i,1,jupv} - \sum_{k\in M}\sum_{t\in M} z_{i,1,jupvkst} = 0;$$
$$i,j,u,v \in M; \quad p,s \in R : 1 < p < s \tag{6}$$

$$y_{i,1,jkst} - \sum_{u\in M}\sum_{v\in M} z_{i,1,jupvkst} = 0;$$
$$i,j,k,t \in M; \quad p,s \in R : 1 < p < s \tag{7}$$

$$y_{i,1,jkst} \;\; - \sum_{p\in R: 1<p<s}\sum_{v\in M} z_{i,1,jupvkst} - \sum_{p\in R: p>s}\sum_{v\in M} z_{i,1,jkstvpu} = 0;$$
$$i,j,k,t,u \in M; \quad s \in R\backslash\{1\} \tag{8}$$

$$y_{upvkst} - \sum_{i\in M}\sum_{j\in M} z_{i,1,jupvkst} = 0;$$
$$u,v,k,t \in M; \quad p,s \in R, \quad 1 < p < s \tag{9}$$

$$\sum_{(k,t)\in M^2} y_{irjkrt} \;\; + \sum_{\substack{(k,t)\in M^2: \\ k\neq j;\ (k,r+1,t)\in A}} y_{irjk,r+1,t} = 0; \quad i,j \in M; \quad r \in R \tag{10}$$

$$\sum_{s\in R:\ s>r}\sum_{k\in M}\sum_{t\in M} y_{irikst} + \sum_{s\in R:\ s<r}\sum_{k\in M}\sum_{t\in M} y_{kstjrj} \sum_{s\in R: s\geq r+1}\sum_{k\in M} y_{irjksi} \;\; +$$
$$+ \sum_{s\in R: s\geq r+1}\sum_{k\in M} y_{irjisk} + \sum_{s\in R: s\geq r+1}\sum_{k\in M} y_{irjksj} + \sum_{s\in R: s\geq r+2}\sum_{k\in M} y_{irjjsk} = 0;$$
$$i,j \in M; \quad r \in R \tag{11}$$

$$y_{irjkst} \in \{0,1\}; \quad i,j,k,\ t \in M; \quad r,s \in R : r < s; \tag{12}$$

$$z_{i,1,jupvkst} \in \{0,1\}; \quad i,\ j,u,\ v,\ k,\ t \in M; \quad p,s \in R : 1 < p < s \tag{13}$$

The propagation of one unit of flow from *stage 1* of *Graph G* is initiated *by* constraint (3). Constraints (4) and (5) ensure that all flows initiated at *stage 1* propagate onward, to *stage n-1* of the graph, in a connected and balanced manner. Specifically, constraints (4) stipulate that the total amount of flow from *arc* $(i, 1, j)$ that propagates through *arc* (k, s, t) and enters *node* $(u, p+1)$ is equal to the amount of flow from *arc* $(i, 1, j)$ that propagates through *arc* (k, s, t) and leaves *node* $(u, p+1)$; Constraints (5) stipulate that the total amount of flow from *arc* $(i, 1, j)$ that enters *node* $(u, p+1)$ to propagate on to *arc* (k, s, t) is equal to the amount of flow from *arc* $(i, 1, j)$ that leaves *node* $(u, p+1)$ to propagate on to *arc* (k, s, t). Constraints (6) and (7) ensure that the propagation of the flow from a given arc at *stage* 1 of *Graph G* onto a given arc at another given stage of the graph is consistently accounted across all the other stages of the graph. Constraints (9) stipulate that the total amount of flow that propagates from *arc* (u, p, v) onto *arc* (k, s, t) is equal to the total of the flows from arcs at *stage* 1 that propagate onto *arc* (k, s, t) via *arc* (u, p, v). Constraints (8) require that the total flow on any given arc of *Graph G* must propagate on to every *level* of the graph, or be part of a flow propagation that spans the *levels* of the graph. Constraints (10) ensure that the initial flow propagation from any given arc occurs in an "unbroken" fashion. Finally, constraints (11) stipulate (in light of the other constraints) that no part of the flow from *arc* (i, r, j) of *Graph G* can propagate back onto neither *level* i nor *level* j of the graph.

Remark 11

1. The number of variables in the system (3)-(11) is $O(n^8)$;
2. The number of constraints in the system (3)-(11) is $O(n^7)$.

Definition 12

1. We refer to the set of points in the space of the y- and z-variables that satisfy the system (3)-(13) as the "IP Polytope," and denote it by Q_I; i.e., $Q_I := \{(y, z) \in \mathcal{R}^\xi : (y, z)\ satisfies$ (3)-(13)$\}$, where ξ is the number of variables in the system (3)-(13).
2. We refer to the linear programming relaxation of Q_I as the "LP Polytope," and denote it by Q_L; i.e., $Q_L := \{(y, z) \in \mathcal{R}^\xi : (y, z)$ satisfies (3)-(11), and $\mathbf{0} \leq (y,\ z) \leq \mathbf{1}\}$, where ξ is the number of variables in the system (3)-(11).

Theorem 13 $(y, z) \in Q_I \iff \exists$ *exactly one set of city indices,* $\{i_r \in M,\ r = 1, \ldots, n-1\}$*, such that:*

$$y_{arbcsd} = \begin{cases} 1 & \textit{for } r, s \in R : r < s, \textit{ and} \\ & (a, b, c, d) = (i_r, i_{r+1}, i_s, i_{s+1}) \\ & \\ 0 & \textit{otherwise} \end{cases} \tag{14}$$

and

$$z_{a,1,bcrdesf} = \begin{cases} 1 & \textit{for } r, s \in R : 1 < r < s, \textit{ and} \\ & (a, b, c, d, e, f) = (i_1, i_2, i_r, i_{r+1}, i_s, i_{s+1}) \\ & \\ 0 & \textit{otherwise} \end{cases} \tag{15}$$

Proof. *a)* $\Longrightarrow$: Let $(y,z) \in Q_I$. Then, given (12), and (13):

Constraint (3) $\Longrightarrow \exists$ a unique set of city indices, $\{i_r \in M,\ r=1,\ldots,4\}$, such that:

$$z_{i_1,1,i_2 i_2,2,i_3 i_3,3,i_4} = 1 \tag{16}$$

Condition (15) follows directly from the combination of (16), (4), and (5).

Condition (14) follows from the combination of Condition (15) with constraints (6)-(7) and (9).

b) $\Longleftarrow$: Trivial. ■

Theorem 14 *There exists a one-to-one correspondence between the points of Q_I and the points of W_I.*

Proof. Combining *Theorem* 13 with constraints (11), (12), and (13), we must have:

$$(y,z) \in Q_I \Longleftrightarrow \exists (i_1, i_2, ..., i_{n-1}) \in M^{n-1}\colon \begin{cases} i) \ \ \text{(14) and (15) are satisfied for } (y,z), \text{ and} \\ \\ ii) \ \ i_p \neq i_q \ \forall (p,q) \in (T, T\backslash\{p\}) \end{cases} \tag{17}$$

Definition 7.4 and Condition (17) $\Longrightarrow$There exists a one-to-one correspondence between the points in Q_I and *c.a.s.s. paths* of *Graph G*. The theorem follows directly from the combination of this and Remark 9.2. ■

Corollary 15 *There exists a one-to-one correspondence between points in Q_I and TSP tours.*

Definition 16

1. We refer to the point $w \in W_I$ correponding to $(y,z) \in Q_I$ as the "assignment corresponding to (y,z)," and denote it by the ordered set $\mathcal{M}(y,z) := \langle i_1, i_2, \cdots, i_{n-1}\rangle$, where $i_q \in M$ is the index of the city visited at time-of-travel q, according to w;

2. We denote the TSP tour corresponding $(y,z) \in Q_I$ as $\mathcal{T}(y,z)$; i.e., for $(y,z) \in Q_I$, $\mathcal{T}(y,z)$ denotes the travel: $1 \longrightarrow i_1 \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_{n-1} \longrightarrow 1$, where $i_q \in \mathcal{M}(y,z)\ \forall q \in T$.

2.1.2 Linear Programming Reformulation

Lemma 17 (Flow conservation lemma 1) *Let $(y,z) \in Q_L$. The following holds true: $\forall (a,b) \in M^2$, $\forall (p,q,r,s) \in R^4 : 1 < p < q;\ 1 < p < r < s$,*

$$\sum_{i_p \in M} \sum_{j_p \in M} \sum_{i_q \in M} \sum_{j_q \in M} z_{a,1,b,i_p,p,j_p,i_q,q,j_q} = \sum_{i_r \in M} \sum_{j_r \in M} \sum_{i_s \in M} \sum_{j_s \in M} z_{a,1,b,i_r,r,j_r,i_s,s,j_s}$$

Proof. $\forall (a,b) \in M^2$, $\forall (p,q,r,s) \in R^4 : 1 < p < q;\ 1 < p < r < s$,

$$\begin{aligned}
\sum_{i_p \in M} \sum_{j_p \in M} \sum_{i_q \in M} \sum_{j_q \in M} z_{a,1,b,i_p,p,j_p,i_q,q,j_q} &= \sum_{i_p \in M} \sum_{j_p \in M} y_{a,1,b,i_p,p,j_p} \quad \text{(Using (6))} \\
&= \sum_{i_p \in M} \sum_{j_p \in M} \sum_{i_r \in M} \sum_{j_r \in M} z_{a,1,b,i_p,p,j_p,i_r,r,j_r} \quad \text{(Using (6))} \\
&= \sum_{i_r \in M} \sum_{j_r \in M} \sum_{i_p \in M} \sum_{j_p \in M} z_{a,1,b,i_p,p,j_p,i_r,r,j_r} \quad \text{(Re-arranging)} \\
&= \sum_{i_r \in M} \sum_{j_r \in M} y_{a,1,b,i_r,r,j_r} \quad \text{(Using (7))} \\
&= \sum_{i_r \in M} \sum_{j_r \in M} \sum_{i_s \in M} \sum_{j_s \in M} z_{a,1,b,i_r,r,j_r,i_s,s,j_s} \quad \text{(Using (6))}
\end{aligned}$$

■

Lemma 18 (Flow propagation lemma 1) *Let $(y,z) \in Q_L$. The following holds true:*

$$\forall (i_1,i_2,i_3,i_4) \in M^4, \quad y_{i_1,1,i_2,i_3,3,i_4} > 0 \iff z_{i_1,1,i_2,i_2,2,i_3,i_3,3,i_4} > 0. \tag{18}$$

Proof. Using constraints (6) and (10), constraints (7) for $p = 2$ and $s = 3$ can be written as:

$$y_{i_1,1,i_2 i_3,3,i_4} - z_{i_1,1,i_2 i_2,2,i_3 i_3,3,i_4} = 0 \quad \forall (i_1,i_2,i_3,i_4) \in M^4 \tag{19}$$

The lemma follows directly from (19). ■

Lemma 19 (Flow propagation lemma 2) *Let $(y,z) \in Q_L$. Then, we must have that:*

$$\forall r \in R : r \geq 4, \quad \forall (i_1,i_2,i_3,i_4,i_r,i_{r+1}) \in M^6,$$

$$z_{i_1,1,i_2,i_3,3,i_4,i_r,r,i_{r+1}} > 0 \Longrightarrow \begin{cases} i)\ z_{i_1,1,i_2,i_2,2,i_3,i_3,3,i_4} > 0; \\ \\ ii)\ z_{i_1,1,i_2,i_2,2,i_3,i_r,r,i_{r+1}} > 0. \end{cases} \tag{20}$$

Proof.

a) Condition i. Using constraints (6),

$$z_{i_1,1,i_2,i_3,3,i_4,i_r,r,i_{r+1}} > 0 \Longrightarrow y_{i_1,1,i_2,i_3,3,i_4} > 0 \quad \forall r \in R : r \geq 4, \quad \forall (i_1,i_2,i_3,i_4,i_r,i_{r+1}) \in M^6 \tag{21}$$

From Lemma 18,

$$y_{i_1,1,i_2,i_3,3,i_4} > 0 \Longrightarrow z_{i_1,1,i_2,i_2,2,i_3,i_3,3,i_4} > 0 \quad \forall (i_1,i_2,i_3,i_4) \in M^4 \tag{22}$$

Condition i) follows directly from (22).

b) Condition ii. Using (6), (7), (9), and (10), constraints (5) for $p = 2$ and $u = i_3$, can be written as:

$$z_{i_1,1,i_2,i_2,2,i_3,i_s,s,i_{s+1}} - \sum_{v \in M} z_{i_1,1,i_2,i_3,3,v,i_s,s,i_{s+1}} = 0 \ \forall (i_1,i_2,i_3,i_s,i_{s+1}) \in M^5; \quad \forall s \in R : s \geq 4 \tag{23}$$

Hence, in particular, we must have:

$$z_{i_1,1,i_2,i_2,2,i_3,i_r,r,i_{r+1}} - z_{i_1,1,i_2,i_3,3,i_4,i_r,r,i_{r+1}} \geq 0 \quad \forall (i_1,i_2,i_3,i_4,i_r,i_{r+1}) \in M^6; \quad \forall\ r \in R : r \geq 4 \tag{24}$$

Condition ii follows directly from (24). ■

Lemma 20 (Flow propagation lemma 3) *The following holds true for all* $(y,z) \in Q_L$*:*

$$\forall r \in R : 2 \le r \le n-4,\ \forall (i_1, i_2, i_r, i_{r+1}, i_{r+2}, i_{r+3}) \in M^6,$$

$$z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+2},r+2,i_{r+3}} > 0 \Longrightarrow \begin{cases} i)\ z_{i_1,1,i_2,i_{r+1},r+1,i_{r+2},i_{r+2},r+2,i_{r+3}} > 0; \\ \\ ii)\ z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+1},r+1,i_{r+2}} > 0. \end{cases} \tag{25}$$

Proof. *a*) *Condition i*. Using (6), (7), (9), and (10), constraints (5) for $p = r$, $s = r+2$, and $u = i_{r+1}$, can be written as:

$$\sum_{v \in M} z_{i_1,1,i_2,v,r,i_{r+1},i_{r+2},r+2,i_{r+3}} - z_{i_1,1,i_2,i_{r+1},r+1,i_{r+2},i_{r+2},r+2,i_{r+3}} = 0$$
$$\forall r \in R : 2 \le r \le n-4,\ \forall (i_1, i_2, i_{r+1}, i_{r+2}, i_{r+3}) \in M^5 \tag{26}$$

Hence, in particular, we must have:

$$z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+2},r+2,i_{r+3}} - z_{i_1,1,i_2,i_{r+1},r+1,i_{r+2},i_{r+2},r+2,i_{r+3}} \le 0$$
$$\forall r \in R : 2 \le r \le n-4,\ \forall (i_1, i_2, i_r, i_{r+1}, i_{r+2}, i_{r+3}) \in M^6 \tag{27}$$

Condition i) follows directly from (27).

b) *Condition ii*. Using constraints (6), (7), (9), and (10), constraints (4) for $p = r+1$, $s = r$, and $u = i_{r+2}$, can be written as:

$$z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+1},r+1,i_{r+2}} - \sum_{v \in M} z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+2},r+2,v} = 0$$
$$\forall r \in R : 2 \le r \le n-4,\ \forall (i_1, i_2, i_{r+1}, i_{r+2}, i_{r+3}) \in M^5 \tag{28}$$

Hence, in particular, we must have:

$$z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+1},r+1,i_{r+2}} - z_{i_1,1,i_2,i_r,r,i_{r+1},i_{r+2},r+2,i_{r+3}} \ge 0$$
$$\forall r \in R : 2 \le r \le n-4,\ \forall (i_1, i_2, i_r, i_{r+1}, i_{r+2}, i_{r+3}) \in M^6 \tag{29}$$

Condition ii) follows directly from (29). ■

Notation 21 *For* $(y,z) \in Q_L$*:*

1. The sub-graph of G induced by the positive components of (y,z) is denoted as:

$$H(y,z) := (P(y,z), E(y,z)), \tag{30}$$

   where:

$$
\begin{aligned}
P(y,z) := & \left\{ (i,1) \in V : \sum_{j \in M} \sum_{t \in M} y_{i,1,jj,2,t} \quad > \quad 0 \right\} \cup \\
& \left\{ (i,r) \in V : \sum_{a \in M} \sum_{b \in M} \sum_{j \in M : (i,r,j) \in A} y_{a,1,birj} \; + \right. \\
& \left. \quad + \sum_{a \in M} \sum_{b \in M} \sum_{j \in M : (j,r-1,i) \in A} y_{a,1,bj,r-1,i} \quad > \quad 0 \right\}
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
E(y,z) := & \left\{ (i,1,j) \in A : \sum_{t \in M} y_{i,1,jj,2,t} \quad > \quad 0 \right\} \cup \\
& \left\{ (i,r,j) \in A : \sum_{a \in M} \sum_{b \in M} \sum_{j \in M : (i,r,j) \in A} y_{a,1,birj} \quad > \quad 0 \right\}.
\end{aligned} \tag{32}
$$

2. The set of arcs of $H(y,z)$ originating at stage r of $H(y,z)$ is denoted $\Gamma_r(y,z)$;

3. The number of arcs originating at stage r of *Graph* $H(y,z)$ is denoted $\gamma_r(y,z) = |\Gamma_r(y,z)|$. For simplicity $\gamma_r(y,z)$ will be henceforth written as γ_r (unless that causes ambiguity);

4. The index set associated with $\Gamma_r(y,z)$ is denoted $\Lambda_r(y,z) := \{1, 2, \ldots, \gamma_r\}$. For simplicity $\Lambda_r(y,z)$ will be henceforth written as Λ_r;

5. The ν^{th} arc in $\Gamma_r(y,z)$ is denoted as $a_{r,\nu}(y,z)$. For simplicity $a_{r,\nu}(y,z)$ will be henceforth written as $a_{r,\nu}$;

6. The tail of $a_{r,\nu}$ is labeled $i_{r,\nu}(y,z)$; the head of $a_{r,\nu}(y,z)$ is labeled $j_{r,\nu}(y,z)$. For simplicity, $i_{r,\nu}(y,z)$ will be henceforth written as $i_{r,\nu}$, and $j_{r,\nu}(y,z)$, as $j_{r,\nu}$;

7. Where that causes no confusion (and where that is convenient), for $r \in R$ with $r \geq 2$, and $(\alpha,\, \rho) \in (\Lambda_1, \Lambda_r)$ "$y_{i_{1,\alpha},1,j_{1,\alpha},i_{r,\rho},r,j_{r,\rho}}$" will be henceforth written as "$y_{(1,\alpha)(r,\rho)}$." Similarly, for $(r,s) \in R^2$ with $1 < r < s$ and $(\alpha, \rho, \sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s)$, "$z_{i_{1,\alpha},1,j_{1,\alpha},i_{r,\rho},r,j_{r,\rho},i_{s,\sigma},s,j_{s,\sigma}}$" will be henceforth written as "$z_{(1,\alpha)(r,\rho)(s,\sigma)}$."

Definition 22 ("*Paths in (y,z)*") *Let* $(y,z) \in Q_L$. $\forall\, (r,s) \in R^2 : s \geq \max\{3, r{+}1\}$, $\forall (\nu_1, \nu_r, \nu_s) \in (\Lambda_1, \Lambda_r, \Lambda_s)$, *a set of arcs of* $H(y,z)$,

$$
\big\{ (a_{r,\nu_r}, \ldots, a_{s,\nu_s}) \in (E(y,z))^{s-r+1} : z_{(1,\nu_1)(p,\nu_p)(q,\nu_q)} > 0 \;\; \forall (p,q) \in R^2 : \\
\max\{2,r\} \leq p < q < s-1; \quad i_{p,\nu_p} = j_{p-1,\nu_{p-1}} \; \forall p \in (R \cap [r+1, s]) \big\}
$$

is referred to as a "path in (y,z) *from* (r,ν_r) *to* (s,ν_s)*."*

Notation 23 *Let* $(y,z) \in Q_L$. $\forall\; (r,s) \in R^2 : s \geq \max\{3, r+1\}$, $\forall\; (\rho,\sigma) \in (\Lambda_r, \Lambda_s)$,

1. The set of all *paths in* (y,z) *from* (r,ρ) *to* (s,σ) is denoted $U_{(r,\rho)(s,\sigma)}(y,z)$;

2. The index set associated with $U_{(r,\rho)(s,\sigma)}(y,z)$ is denoted $\Phi_{(r,\rho)(s,\sigma)}(y,z) :=$ $\{1, 2, \ldots, \varphi_{(r,\rho)(s,\sigma)}(y,z)\}$, where $\varphi_{(r,\rho)(s,\sigma)}(y,z) := \left|U_{(r,\rho)(s,\sigma)}(y,z)\right|$;

3. The k^{th} element of $U_{(r,\rho)(s,\sigma)}(y,z)$ $(k \in \Phi_{(r,\rho)(s,\sigma)}(y,z))$ is denoted $\mathcal{L}_{(r,\rho),(s,\sigma),k}(y,z)$.

Theorem 24 (Distinct *paths in (y,z)*) *Let* $(y,z) \in Q_L$. *Then,* $\forall$ $(r,s) \in R^2 : s \geq \max\{3, r+1\}$, $\forall(\alpha_1,\alpha_2) \in \Lambda_r^2$, $\forall(\beta_1,\beta_2) \in \Lambda_s^2$: $(U_{(r,\alpha_1)(s,\beta_1)}(y,z) \neq \emptyset$ *and* $U_{(r,\alpha_2)(s,\beta_2)}(y,z) \neq \emptyset)$, $\forall$ $k \in \Phi_{(r,\alpha_1)(s,\beta_1)}(y,z)$, $\forall$ $t \in \Phi_{(r,\alpha_2)(s,\beta_2)}(y,z)$, $\quad \mathcal{L}_{(r,\alpha_1),(s,\beta_1),k}(y,z) \neq \mathcal{L}_{(r,\alpha_2),(s,\beta_2),t}(y,z) \iff \exists$ $\{g \in R : r \leq g \leq s;$
$(\gamma_1,\gamma_2) \in (\Lambda_g, \Lambda_g \backslash \{\gamma_1\})\} \ni \left\{a_{g,\gamma_1} \in \mathcal{L}_{(r,\alpha_1),(s,\beta_1),k}(y,z) \text{ and } a_{g,\gamma_2} \in \mathcal{L}_{(r,\alpha_2),(s,\beta_2),t}(y,z)\right\}$.

Proof. The theorem follows directly from the combination of constraints (6), (7), (9), and (10), and Definition 22. ■

Theorem 25 (Path structure theorem 1) *Let* $(y,z) \in Q_L$. *The following holds true:*
$\forall(r,s) \in R^2$ *with* $s \geq \max\{3, r+1\}$, $\forall(\rho,\sigma) \in (\Lambda_r, \Lambda_s)$, $y_{(r,\rho)(s,\sigma)} > 0 \iff U_{(r,\rho)(s,\sigma)}(y,z) \neq \emptyset$.

Proof. First, note that it follows directly from the combination of Lemmas 18, 19, and 20, that the theorem holds true for all $(r,s) \in R^2$ with $s \in \{r+1, r+2\}$, and all $(\nu_r, \nu_s) \in (\Lambda_r, \Lambda_s)$.

a) $\Longrightarrow$: Assume there exists an integer $\omega \geq 2$ such that the theorem holds true for all $(p,t) \in R^2$ with $t = p + \omega$, and all $(\nu_p, \nu_t) \in (\Lambda_p, \Lambda_t)$. We will show that the theorem must hold for all $(p,u) \in R^2$ with $u = t + 1 = p + \omega + 1$, and all $(\nu_p, \nu_u) \in (\Lambda_r, \Lambda_u)$.

Let $(p,u) \in R^2$ with $u = p + \omega + 1$, and $(\nu_p, \nu_u) \in (\Lambda_p, \Lambda_u)$ be such that:

$$y_{(p,\nu_p)(u,\nu_u)} > 0. \tag{33}$$

Define:

$$B_{(p,\nu_p)(u,\nu_u)}(y,z) := \{\alpha \in \Lambda_1 : z_{(1,\alpha)(p,\nu_p)(u,\nu_u)} > 0\}. \tag{34}$$

Then, (6), (7) and (33)$\Longrightarrow$

$$\begin{cases} i) \;\; B_{(p,\nu_p)(u,\nu_u)}(y,z) \neq \varnothing, \quad \text{with} \\ ii) \;\; y_{(p,\nu_p)(u,\nu_u)} = \displaystyle\sum_{\alpha \in B_{(p,\nu_p)(u,\nu_u)}(y,z)} z_{(1,\alpha)(p,\nu_p)(u,\nu_u)} \end{cases} \tag{35}$$

Condition (35), and constraints (5) and (10) $\Longrightarrow$

$$\forall \; \alpha \in B_{(p,\nu_p)(u,\nu_u)}(y,z), \;\; \exists \; C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z) \subseteq \Lambda_{p+1} \quad \ni \; :$$

$$\begin{cases} i) \;\; i_{p+1,\beta} = j_{p,\nu_p} \;\; \forall \; \beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z) \\ ii) \;\; z_{(1,\alpha)(p+1,\beta)(u,\nu_u)} > 0 \quad \forall \; \beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z); \text{ and} \\ iii) \;\; z_{(1,\alpha)(p,\nu_p)(u,\nu_u)} \leq \displaystyle\sum_{\beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z)} z_{(1,\alpha)(p+1,\beta)(u,\nu_u)}. \end{cases} \tag{36}$$

By assumption (since $u = (p+1) + \omega$), condition (36.*ii*) $\Longrightarrow$

$$U_{(p+1,\beta)(u,\nu_u)}(y,z) \neq \emptyset \;\; \forall \beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z) \tag{37}$$

Also, it follows from the combination of condition (33), constraints (4), and constraints (8), that:

$$\begin{aligned} &\forall\ \beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z),\ \exists\ \left\{\Upsilon_{(p,\nu_p)(p+1,\beta)(u,\nu_u)}(y,z) \subseteq \Phi_{(p+1,\beta)(u,\nu_u)}(y,z)\right\} \ \ni\ : \\ &\left\{(z)_{(1,\alpha)(p,\nu_p)(q,\nu_{q,\iota})} > 0 \quad \forall \iota \in \Upsilon_{(p,\nu_p)(p+1,\beta)(u,\nu_u)}(y,z),\right. \\ &\left.\forall q \in (R \cap [p+1,u]),\ \text{and}\ \forall\ \nu_{q,\iota} \in \Lambda_q : a_{p,\nu_{p,\iota}} \in\ \mathcal{L}_{(p+1,\beta)(u,\nu_u),\iota}(y,z)\right\} \end{aligned} \tag{38}$$

Hence, $\forall\ \beta \in C_{\alpha,(p,\nu_p)(u,\nu_u)}(y,z)$ and $\forall\ \iota \in \Upsilon_{(p,\nu_p)(p+1,\beta)(u,\nu_u)}(y,z)$,

$$\overline{L} := \left(\mathcal{L}_{(p+1,\beta)(u,\nu_u),\iota}(y,z) \quad \cup \quad \{a_{p,\nu_p}\}\right)$$

is a *path in (y,z) from* (p,ν_p) *to* (u,ν_u). Hence, we have that $U_{(p,\nu_p)(u,\nu_u)}(y,z) \neq \emptyset$.

b) $\Longleftarrow$: Follows directly from Definition (22) and constraints (9). ■

Corollary 26 *Let* $(y,z) \in Q_L$. *The following hold true:*

i) $\forall\ s \in R\backslash\{1\}$, $\forall(\alpha,\sigma) \in (\Lambda_1,\Lambda_s)$, $y_{(1,\alpha)(s,\sigma)} > 0 \Longleftrightarrow U_{(1,\alpha)(s,\sigma)}(y,z) \neq \emptyset$.

ii) $\forall\ (r,s) \in (R\backslash\{1\})^2$ *with* $s \geq \max\{3, r+1\}$, $\forall(\alpha,\rho,\sigma) \in (\Lambda_1,\Lambda_r,\Lambda_s)$,

$$z_{(1,\alpha)(r,\rho)(s,\sigma)} > 0 \Longleftrightarrow \begin{cases} ii.1) \;\; U_{(1,\alpha)(s,\sigma)}(y,z) \neq \emptyset, \quad \textit{and} \\ \\ ii.2) \;\; \exists\ \kappa \in \Phi_{(1,\alpha)(s,\sigma)}(y,z) \ \ni\ a_{r,\rho} \in \mathcal{L}_{(1,\alpha),(s,\sigma),\kappa}(y,z). \end{cases}$$

Definition 27 (*"TSP tour in (y,z)"*) *Let* $(y,z) \in Q_L$. $\forall(\nu_1,\nu_{n-2}) \in (\Lambda_1,\Lambda_{n-2})$, *a* path in (y,z) from $(1,\nu_1)$ to $(n-2,\nu_{n-2})$ *is referred to as a "TSP tour in* (y,z) (from $(1,\nu_1)$ to (n,ν_{n-2}))*."*

Notation 28 *Let* $(y,z) \in Q_L$. *For all* $(\alpha,\beta) \in (\Lambda_1,\Lambda_{n-2})$,

1. The set of all *paths in* (y,z) *from* $(1,\alpha)$ *to* $(n-2,\beta)$ is denoted as $\Pi_{\alpha\beta}(y,z)$;
2. The index set associated with $\Pi_{\alpha\beta}(y,z)$ is denoted $\Psi_{\alpha\beta}(y,z) := \{1,\ 2,\ \ldots,\ \pi_{\alpha\beta}(y,z)\}$, where $\pi_{\alpha\beta}(y,z) := |\Pi_{\alpha\beta}(y,z)|$;
3. The k^{th} element of $\Pi_{\alpha\beta}(y,z)$ is denoted $\mathcal{P}_{\alpha\beta k}(y,z)$.

Remark 29 *Let* $(y,z) \in Q_L$. $\forall\ (\alpha,\beta) \in (\Lambda_1,\Lambda_{n-2})$,

1. $\Pi_{\alpha\beta}(y,z) = U_{(1,\alpha)(n-2,\beta)}(y,z)$;
2. $\Psi_{\alpha\beta}(y,z) = \Phi_{(1,\alpha),(n-2,\beta)}(y,z)$;
3. $\pi_{\alpha\beta}(y,z) = \varphi_{(1,\alpha)(n-2,\beta)}(y,z)$;
4. We assume (w.l.o.g.) that: $\mathcal{P}_{\alpha\beta k}(y,z) = \mathcal{L}_{(1,\alpha),(n-2,\beta),k}(y,z) \;\; \forall\ k \in \varphi_{\alpha\beta}(y,z)$.

Theorem 30 (Equivalences for TSP tours in (y,z)) *For* $(y,z) \in Q_L$:

i) every *TSP tour in* (y, z) corresponds to exactly one perfect matching of M and T;

ii) every *TSP tour in* (y, z) corresponds to exactly one TSP tour.

Proof. Definition 22, constraints (11), and Remark 8, and Definitions 7.3-7.4 imply that every *TSP tour in* (y, z) corresponds to exactly one *c.a.s.s. path* of *Graph G*. The theorem follows directly from the combination of this and Remark 9. ∎

Theorem 31 ("Convex independence" of TSP tours in (y,z)) *Let* $(y, z) \in Q_L$. *A given TSP tour in* (y, z) *cannot be represented as a convex combination of other TSP tours in* (y, z).

Proof. Theorem 30 implies that every TSP $tour$ in (y, z) coresponds to an extreme point of $Conv(W_L) = Conv(W_I)$. The theorem follows directly from this. ∎

Theorem 32 (Path structure theorem 2) *Let* $(y, z) \in Q_L$. *The following holds true:* $\forall\ r \in R$, $\forall\ \rho \in \Lambda_r$, $\exists\ \{(\alpha, \beta) \in (\Lambda_1, \Lambda_{n-2});\ \iota \in \Psi_{\alpha\beta}(y, z)\} \ni \{a_{r,\rho} \in \mathcal{P}_{\alpha\beta\iota}(y, z)\}$.

Proof.

Case 1: r = 1. From (31) and (32):

$$a_{r,\rho} \in E(y, z) \Longrightarrow \exists\ \alpha\ \in\ \Lambda_2\ \ni\ y_{(r,\rho)(2,\alpha)} > 0. \tag{39}$$

Condition (39) and constraints (6) $\Longrightarrow$

$$\exists\ \gamma \in \Lambda_{n-2}\ \ni\ z_{(r,\rho)(2,\alpha)(n-2,\gamma)} > 0. \tag{40}$$

Condition (40) and constraints (6) $\Longrightarrow$

$$\exists\ \gamma \in \Lambda_{n-2}\ \ni\ y_{(r,\rho)(n-2,\gamma)} > 0. \tag{41}$$

The theorem follows from the combination of (41) with Theorem 25.

Case 2: r = n-2. From (31) and (32):

$$a_{r,\rho} \in E(y, z) \Longrightarrow \exists\ \alpha\ \in\ \Lambda_1\ \ni\ y_{(1,\alpha)(r,\rho)} > 0. \tag{42}$$

The theorem follows from the combination of (42) with Theorem 25.

Case 3: 1 < r < n-2. From (31) and (32):

$$a_{r,\rho} \in E(y, z) \Longrightarrow \exists\ \alpha\ \in\ \Lambda_1\ \ni\ y_{(1,\alpha)(r,\rho)} > 0. \tag{43}$$

Condition (43) and constraints (6) $\Longrightarrow$

$$\exists\gamma \in \Lambda_{n-2}\ \ni\ z_{(1,\alpha)(r,\rho)(n-2,\gamma)} > 0. \tag{44}$$

The theorem follows from the combination of (44) with Corollary 26*ii*. ∎

Corollary 33 *Let $(y,z) \in Q_L$. The following hold true:*

i) $\forall \alpha \in \Lambda_1$, $\forall r \in R\backslash\{1\}$, $\forall \rho \in \Lambda_r$,

$$y_{(1,\alpha)(r,\rho)} > 0 \Longleftrightarrow \exists \ (\beta \in \Lambda_{n-2};\ \iota \in \Psi_{\alpha\beta}(y,z)) \ni \quad a_{r,\rho} \in \mathcal{P}_{\alpha\beta\iota}(y,z); \tag{45}$$

ii) $\forall (r,s) \in R^2 : 1 < r < s$, $\forall (\alpha,\rho,\sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s)$,

$$z_{(1,\alpha)(r,\rho)(s,\sigma)} > 0 \Longleftrightarrow \exists \ (\beta \in \Lambda_{n-2};\ \iota \in \Psi_{\alpha\beta}(y,z)) \ni \quad (a_{r,\rho}, a_{s,\sigma}) \in \mathcal{P}^2_{\alpha\beta\iota}(y,z). \tag{46}$$

Lemma 34 (Flow conservation lemma 2) *Let $(y,z) \in Q_L$. The following hold true:*

$$i)\ y_{(1,\alpha)(r,\nu_r)} = \sum_{\nu_p \in \Lambda_p} \sum_{\beta \in \Lambda_{n-2}} \sum_{\substack{\iota \in \Psi_{\alpha\beta}(y,z): \\ a_{p,\nu_p} \in \mathcal{P}_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(p,\nu_p)(r,\nu_r)}$$

$$\forall \alpha \in \Lambda_1,\ \forall (p,r) \in R^2 : 1 < p < r,\ \forall\ \nu_r \in \Lambda_r$$

$$ii)\ y_{(1,\alpha)(r,\nu_r)} = \sum_{\nu_q \in \Lambda_q} \sum_{\beta \in \Lambda_{n-2}} \sum_{\substack{\iota \in \Psi_{\alpha\beta}(y,z): \\ a_{q,\nu_q} \in \mathcal{P}_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(r,\nu_r)(q,\nu_q)}$$

$$\forall \alpha \in \Lambda_1,\ \forall (r,q) \in R^2 : 1 < r < q,\ \forall\ \nu_r \in \Lambda_r$$

Proof. The lemma follows directly from the combination of constraints (6) and (7), and Theorems 24, 31, and 32, and Corollary 33. ■

Definition 35 ("Weights" of TSP tours in (y,z)) *Let $(y,z) \in Q_L$. For $(\alpha,\beta) \in (\Lambda_1, \Lambda_{n-2})$ such that $y_{(1,\alpha)(n-2,\beta)} > 0$, and $k \in \Psi_{1,n-2}(y,z)$, we refer to the quantity*

$$\omega_{\alpha\beta k}(y,z) := \min_{\substack{(p,q) \in R^2;\ (\nu_p,\nu_q)\ \in\ (\Lambda_p,\Lambda_q): \\ 1<p<q;\ (a_{p,\nu_p},\ a_{q,\nu_q})\ \in\ P^2_{\alpha\beta k}(y,z)}} \left\{ z_{(1,\alpha)(p,\nu_p)(q,\nu_q)} \right\} \tag{47}$$

as the "weight" of (TSP tour in (y,z)) $\mathcal{P}_{\alpha\beta k}(y,z)$.

Remark 36 *It follows directly from Definitions 22 and 35 that for $(y,z) \in Q_L$, $\omega_{\alpha\beta\iota}(y,z) > 0$ $\forall$ $(\alpha,\beta) \in (\Lambda_1, \Lambda_{n-2}) : \Psi_{\alpha\beta}(y,z) \neq \varnothing$, $\forall\ \iota \in P_{\alpha\beta\iota}(y,z)$.*

Theorem 37 (Path structure theorem 3) *Let $(y,z) \in Q_L$. The following hold true:*

i) $\forall\ \alpha \in \Lambda_1$, $\forall\ r \in R\backslash\{1\}$, $\forall\ \rho \in \Lambda_r$,

$$y_{(1,\alpha)(r,\rho)} = \sum_{\beta \in \Lambda_{n-2}} \sum_{\substack{\iota \in \Psi_{\alpha\beta}(y,z): \\ a_{r,\rho}\ \in\ \mathcal{P}_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

ii) $\forall (r,s) \in R^2 : 1 < r < s$, $\forall (\alpha,\rho,\sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s)$,

$$z_{(1,\alpha)(r,\rho)(s,\sigma)} = \sum_{\beta \in \Lambda_{n-2}} \sum_{\substack{\iota \in \Psi_{\alpha\beta}(y,z): \\ (a_{r,\rho},\ a_{s\sigma})\ \in\ \mathcal{P}^2_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

iii) $\forall (r,s) \in R^2 : 1 < r < s$, $\forall (\rho,\sigma) \in (\Lambda_r, \Lambda_s)$,

$$y_{(r,\rho)(s,\sigma)} = \sum_{\alpha\in\Lambda_1}\sum_{\beta\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\rho},\ a_{s,\sigma})\,\in\,\mathcal{P}^2_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

Proof. *a) Condition i*. First, note that from the combination of constraints (3), (4), (5), and (10); Remark 36; and Theorems (31), and (32), we must have:

$$\sum_{\alpha\in\Lambda_1}\sum_{\substack{\beta\in\Lambda_2:\\ i_{2,\beta}=j_{1,\alpha}}}\sum_{\substack{\delta\in\Lambda_3:\\ i_{3,\delta}=j_{2,\beta}}} z_{(1,\alpha)(2,\beta)(3,\delta)} = \sum_{\alpha\in\Lambda_1}\sum_{\varrho\in\Lambda_{n-2}}\sum_{\iota\in\Psi_{\alpha\varrho}(y,z)} \omega_{\alpha\varrho\iota}(y,z) = 1. \tag{48}$$

a.1) From Definition 35, (48) $\Longrightarrow$

$$\sum_{\substack{\beta\in\Lambda_2:\\ i_{2,\beta}=j_{1,\alpha}}}\sum_{\substack{\delta\in\Lambda_3:\\ i_{3,\delta}=j_{2,\beta}}} z_{(1,\alpha)(2,\beta)(3,\delta)} = \sum_{\varrho\in\Lambda_{n-2}}\sum_{\iota\in\Psi_{\alpha\varrho}(y,z)} \omega_{\alpha\varrho\iota}(y,z) \quad \forall \alpha\in\Lambda_1 \tag{49}$$

Lemma 17 and relations (49) $\Longrightarrow$

$$\sum_{\rho\in\Lambda_r}\sum_{\sigma\in\Lambda_s} z_{(1,\alpha)(r,\rho)(s,\sigma)} = \sum_{\varrho\in\Lambda_{n-2}}\sum_{\iota\in\Psi_{\alpha\varrho}(y,z)} \omega_{\alpha\varrho\iota}(y,z) \quad \forall\alpha\in\Lambda_1, \forall (r,s)\in R^2 : 1<r<s \tag{50}$$

Using constraints (6), relations (50) $\Longrightarrow$:

$$\sum_{\rho\in\Lambda_r} y_{(1,\alpha)(r,\rho)} = \sum_{\varrho\in\Lambda_{n-2}}\sum_{\iota\in\Psi_{\alpha\varrho}(y,z)} \omega_{\alpha\varrho\iota}(y,z) \quad \forall\alpha\in\Lambda_1, \forall r\in R\backslash\{1,n-2\} \tag{51}$$

Using Theorem 24, (51) can be written as:

$$\sum_{\rho\in\Lambda_r} y_{(1,\alpha)(r,\rho)} = \sum_{\rho\in\Lambda_r}\sum_{\varrho\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{r,\rho}\,\in\,\mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \quad \forall\alpha\in\Lambda_1, \forall r\in R\backslash\{1,n-2\} \tag{52}$$

Re-arranging (52) gives:

$$\sum_{\rho\in\Lambda_r}\left(y_{(1,\alpha)(r,\rho)} - \sum_{\varrho\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{r,\rho}\,\in\,\mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z)\right) = 0 \quad \forall\alpha\in\Lambda_1, \forall r\in R\backslash\{1,n-2\} \tag{53}$$

a.2) Combining Lemma 34.*ii* with Definition 35, we have that:

$$\begin{aligned} y_{(1,\alpha)(r,\rho)} &= \sum_{\nu_q\in\Lambda_q}\sum_{\varrho\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{q,\nu_q}\,\in\,\mathcal{P}_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(r,\rho)(q,\nu_q)} \\ &\geq \sum_{\nu_q\in\Lambda_q}\sum_{\varrho\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ (a_{r,\rho},\ a_{q,\nu_q})\,\in\,\mathcal{P}^2_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(r,\rho)(q,\nu_q)} \\ &\geq \sum_{\varrho\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\varrho}(y,z):\\ a_{r,\rho}\,\in\,\mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \end{aligned}$$

$$\forall\ r\in R\backslash\{1,n-2\},\ \forall q\in R: q>r,\ \forall (\alpha,\rho)\in(\Lambda_1,\Lambda_r) \tag{54}$$

Relations (53) and (54) $\Longrightarrow$

$$y_{(1,\alpha)(r,\rho)} = \sum_{\varrho \in \Lambda_{n-2}} \sum_{\substack{\iota \in \Psi_{\alpha\varrho}(y,z): \\ a_{r,\rho} \in \mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \quad \forall r \in R \backslash \{1, n-2\}, \forall (\alpha, \rho) \in (\Lambda_1, \Lambda_r) \tag{55}$$

$a.3$) Using constraints (7), relations (50) $\Longrightarrow$:

$$\sum_{\sigma \in \Lambda_s} y_{(1,\alpha)(s,\sigma)} = \sum_{\varrho \in \Lambda_{n-2}} \sum_{\iota \in \Psi_{\alpha\varrho}(y,z)} \omega_{\alpha\varrho\iota}(y,z) \quad \forall \alpha \in \Lambda_1, \forall s \in R : s > 2 \tag{56}$$

Using Theorem 24, (56) $\Longrightarrow$

$$\sum_{\sigma \in \Lambda_s} y_{(1,\alpha)(s,\sigma)} = \sum_{\sigma \in \Lambda_s} \sum_{\varrho \in \Lambda_{n-2}} \sum_{\substack{\iota \in \Psi_{\alpha\varrho}(y,z): \\ a_{s,\sigma} \in \mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \quad \forall \alpha \in \Lambda_1, \forall s \in R : s > 2 \tag{57}$$

Re-arranging (57) gives:

$$\sum_{\sigma \in \Lambda_s} \left(y_{(1,\alpha)(s,\sigma)} - \sum_{\varrho \in \Lambda_{n-2}} \sum_{\substack{\iota \in \Psi_{\alpha\varrho}(y,z): \\ a_{s,\sigma} \in \mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \right) = 0 \quad \forall \alpha \in \Lambda_1, \forall s \in R : s > 2 \tag{58}$$

$a.4$) Combining Lemma 34.i with Definition 35, we have that:

$$\begin{aligned} y_{(1,\alpha)(s,\sigma)} &= \sum_{\nu_p \in \Lambda_p} \sum_{\varrho \in \Lambda_{n-2}} \sum_{\substack{\iota \in \Psi_{\alpha\varrho}(y,z): \\ a_{p,\nu_p} \in \mathcal{P}_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(p,\nu_p)(s,\sigma)} \\ &\geq \sum_{\nu_p \in \Lambda_p} \sum_{\varrho \in \Lambda_{n-2}} \sum_{\substack{\iota \in \Psi_{\alpha\varrho}(y,z): \\ (a_{p,\nu_p},\, a_{s,\nu_s}) \in \mathcal{P}^2_{\alpha\beta\iota}(y,z)}} z_{(1,\alpha)(p,\nu_p)(s,\sigma)} \\ &\geq \sum_{\varrho \in \Lambda_{n-2}} \sum_{\substack{\iota \in \Psi_{\alpha\varrho}(y,z): \\ a_{s,\sigma} \in \mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \end{aligned}$$

$$\forall (p,s) \in R^2 : 1 < p < s;\ s > 2,\ \forall (\alpha, \sigma) \in (\Lambda_1, \Lambda_s) \tag{59}$$

$a.5$) Relations (58) and (59) $\Longrightarrow$

$$y_{(1,\alpha)(s,\sigma)} = \sum_{\varrho \in \Lambda_{n-2}} \sum_{\substack{\iota \in \Psi_{\alpha\varrho}(y,z): \\ a_{s,\sigma} \in \mathcal{P}_{\alpha\varrho\iota}(y,z)}} \omega_{\alpha\varrho\iota}(y,z) \quad \forall \alpha \in \Lambda_1, \forall s \in R : s > 2 \tag{60}$$

$a.6$) *Condition* i of the theorem follows from the combination of (55) and (60).

b) *Condition* ii.

$b.1)$ Using Theorem 24 and Corollary 33.ii, (50) can be re-written as:

$$\sum_{\rho\in\Lambda_r}\sum_{\sigma\in\Lambda_s}\sum_{\beta\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\(a_{r,\rho},\ a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}}\omega_{\alpha\beta\iota}(y,z)=\sum_{\rho\in\Lambda_r}\sum_{\sigma\in\Lambda_s}z_{(1,\alpha)(r,\rho)(s,\sigma)}$$

$$\forall(r,s)\in R^2:1<r<s,\ \forall\alpha\in\Lambda_1 \tag{61}$$

Re-arranging (61) gives:

$$\sum_{\rho\in\Lambda_r}\sum_{\sigma\in\Lambda_s}\left(z_{(1,\alpha)(r,\rho)(s,\sigma)}-\sum_{\beta\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\(a_{r,\rho},\ a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}}\omega_{\alpha\beta\iota}(y,z)\right)=0$$

$$\forall(r,s)\in R^2:1<r<s,\ \forall\alpha\in\Lambda_1 \tag{62}$$

$b.2)$ From Lemma 34.ii, we have:

$$y_{(1,\alpha)(s,\sigma)}=z_{(1,\alpha)(r,\rho)(s,\sigma)}+\sum_{\substack{\nu_r\in\Lambda_r:\\\nu_r\neq\rho}}\sum_{\beta\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\(a_{r,\nu_r},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}}z_{(1,\alpha)(r,\nu_r)(s,\sigma)}$$

$$\forall(r,s)\in R^2:1<r<s,\ \forall(\alpha,\rho,\sigma)\in(\Lambda_1,\Lambda_r,\Lambda_s). \tag{63}$$

$b.3)$ From *Condition i*, we have:

$$y_{(1,\alpha)(s,\sigma)}=\sum_{\beta\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\(a_{r,\rho},\ a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}}\omega_{\alpha\beta\iota}(y,z)\ +\sum_{\substack{\nu_r\in\Lambda_r:\\\nu_r\neq\rho}}\sum_{\beta\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\(a_{r,\nu_r},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}}\omega_{\alpha\beta\iota}(y,z)$$

$$\forall(r,s)\in R^2:1<r<s,\ \forall(\alpha,\rho,\sigma)\in(\Lambda_1,\Lambda_r,\Lambda_s) \tag{64}$$

$b.4)$ Definition 35 $\Longrightarrow$:

$$\sum_{\substack{\nu_r\in\Lambda_r:\\\nu_r\neq\rho}}\sum_{\beta\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\(a_{r,\nu_r},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}}z_{(1,\alpha)(r,\nu_r)(s,\sigma)}\geq\sum_{\substack{\nu_r\in\Lambda_r:\\\nu_r\neq\rho}}\sum_{\beta\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\(a_{r,\nu_r},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}}\omega_{\alpha\beta\iota}(y,z)$$

$$\forall(r,s)\in R^2:1<r<s,\ \forall(\alpha,\rho,\sigma)\in(\Lambda_1,\Lambda_r,\Lambda_s) \tag{65}$$

$b.5)$ Relations (63)-(65) $\Longrightarrow$

$$z_{(1,\alpha)(r,\rho)(s,\sigma)}\leq\sum_{\beta\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\(a_{r,\rho},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}}\omega_{\alpha\beta\iota}(y,z)$$

$$\forall(r,s)\in R^2:1<r<s,\ \forall(\alpha,\rho,\sigma)\in(\Lambda_1,\Lambda_r,\Lambda_s) \tag{66}$$

$b.6)$ Combining (62) and (66), we must have:

$$z_{(1,\alpha)(r,\rho)(s,\nu_s)}=\sum_{\beta\in\Lambda_{n-2}}\sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\(a_{r,\rho},a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}}\omega_{\alpha\beta\iota}(y,z)$$

$$\forall(r,s)\in R^2:1<r<s,\ \forall(\alpha,\rho,\sigma)\in(\Lambda_1,\Lambda_r,\Lambda_s) \tag{67}$$

c) Condition iii. From the combination of constraints (9) and *Condition ii*, we have:

$$y_{(r,\rho)(s,\sigma)} = \sum_{\alpha\in\Lambda_1} z_{(1,\alpha)(r,\rho)(s,\sigma)} = \sum_{\alpha\in\Lambda_1}\sum_{\beta\in\Lambda_{n-2}} \sum_{\substack{\iota\in\Psi_{\alpha\beta}(y,z):\\ (a_{r,\rho},\ a_{s,\sigma})\in P^2_{\alpha\beta\iota}(y,z)}} \omega_{\alpha\beta\iota}(y,z)$$

$$\forall(r,s)\in R^2 : 1<r<s,\ \forall(\rho,\sigma)\in(\Lambda_r,\Lambda_s). \tag{68}$$

■

Corollary 38 $(y,\,z)\in Q_L \Longleftrightarrow (y,z)$ *corresponds to a convex combination of perfect matchings of* M *and* T *with coefficients equal to the weights of the corresponding TSP tours in (y,z).*

Theorem 39 *The following holds true:* $Conv(Q_L) = Conv(Q_I)$.

Proof. The theorem follows directly from the combination of Theorem 30, Theorem 31, and Corollary 38. ■

Corollary 40 *The following mappings are bijective:*

1. $\mathcal{B}_1 : Conv(Q_L) \longmapsto Conv(W_I)$;
2. $\mathcal{B}_2 : Conv(Q_L) \longmapsto Conv(W_L)$;
3. $\mathcal{B}_3 : Ext(Q_L) \longmapsto Ext(Conv(W_I))$;
4. $\mathcal{B}_4 : Ext(Q_L) \longmapsto Ext(W_L)$;
5. $\mathcal{B}_5 : Ext(Q_L) \longmapsto \Omega$;
6. $\mathcal{B}_6 : Ext(Q_L) \longmapsto \Delta$.

2.2 Reformulation of the Travel Costs

Definition 41 (Re-defined costs) *Let* $(y,z)\in Q_L$. $\forall(i,j,u,v,k,t)\in M^6$, $\forall(r,s)\in R^2 : 1 < r < s$, *the "cost" associated with* $z_{u1,virjkst}$ *is defined as:*

$$c_{u1,virjkst} := \begin{cases} x_{1,u}+x_{u,v}+x_{v,j}+x_{j,t} & \text{if } \ r=2;\ \ s=3;\ \ i=v;\ \ k=j \\[2ex] x_{j,t} & \text{if } \ r=2;\ \ 4\le s\le n-3;\ \ i=v \\[2ex] x_{j,t}+x_{t,1} & \text{if } \ r=2;\ \ s=n-2;\ \ i=v \\[2ex] 0 \ \ \text{Otherwise} \end{cases}$$

Theorem 42 *Let:*

$$\vartheta(y,z) := c^T\cdot z \ +\ \mathbf{0}^T\cdot y = \sum_{u\in M}\sum_{v\in M}\sum_{i\in M}\sum_{\substack{r\in R:\\ r>1}}\sum_{j\in M}\sum_{k\in M}\sum_{\substack{s\in R:\\ s>r}}\sum_{t\in M} c_{u,1,virjkst}z_{u,1,virjkst}. \tag{69}$$

Then, for $(y,z)\in Ext(Q_L)$, $\vartheta(y,z)$ *accurately accounts the cost of the TSP corresponding to (y, z).*

Proof. From Theorem 39,

$$(y,z) \in Ext(Q_L) \Longleftrightarrow (y,z) \in Q_I \tag{70}$$

Now, using Theorems 13, it can be verified directly that for $(y,z) \in Q_I$,

$$\vartheta(y,z) = x_{1,i_1} + \sum_{r=1}^{n-2} x_{i_r,i_{r+1}} + x_{i_{n-1},1}, \quad \text{where } i_r \in \mathcal{M}(y,z) \quad \forall\, r \in T. \ \blacksquare$$

2.3 Overall Linear Program

Our overall linear programming model is as follows:

Problem 43 (*Problem LP*)

$$min\ \{\vartheta(y,z) : (y,z) \in Q_L\}$$

Theorem 44 *The following statements are true of basic feasible solutions (BFS) of Problem LP and TSP tours:*

1. Every BFS of *Problem LP* corresponds to a TSP tour;
2. Every TSP tour corresponds to a BFS of *Problem LP*;
3. The mapping of BFS's of *Problem LP* onto TSP tours is surjective.

Proof. Statements (44.1) and (44.2) follow directly from the combination of Theorem 39, Corollary 40.6, and the correspondence between BFS's of LP models and extreme points of their associated polyhedra (see [1, pp. 92-101]). Statement (44.3) follows from the primal degeneracy of *Problem LP* (see [8, p. 32]). $\blacksquare$

Corollary 45 *Problem LP solves the TSP.*

3 Numerical Implementation

We implemented a streamlined version of *Problem LP* in which constraints (9)-(11) were handled implicitly, and the upper bounds on the variables were omitted. We solved 3 problems with $n = 8$ cities, and 3 problems with $n = 9$ cities. One of the 8-city problems and one of the 9-city problems had, each, travel costs all equal to zero (i.e., we had $t_{ij} = 0\ \forall (i,j) \in (N, N\backslash\{i\}$ in these problems). Each of the remaining four of our test problems had travel costs that were randomly generated between 1 and 300. Also, for each problem size, one of the randomly-generated problems had symmetric travel costs, while the other had asymmetric costs.

We used the simplex implementation of the "Clp" routines of the *COIN-OR* open source library ([2]) to solve the primal and dual forms of each of the test problems, respectively. The computational results are summarized in Table 1. In general, the dual forms performed better than the primal forms. The computational times across the two forms range from 1.02 seconds to 34.50 seconds (Sony VAIO notebook computer with 1.83 GHz Intel Core 2 Duo processor) for the 8-city problems, and from 33.84 seconds to 4,818.44 seconds for the 9-city problems. Comparable times are on the order of several hours (more than 6, in general) for 8-city problems for the model in [3] .

Problem Number	Number of cities	Type of cost[a]	Primal Form		Dual Form	
			# of iterations[b]	CPU seconds[c]	# of iterations[b]	CPU seconds[c]
1	8	A	15,354	11.64	1,274	1.02
2	8	S	11,227	9.23	443	0.56
3	8	N	10,265	2.84	10,610	34.50
4	9	A	149,257	2,653.92	3,145	6.77
5	9	S	217,914	4,818.44	70,598	384.25
6	9	N	32,831	33.84	69,498	2,149.55

a: "A" = asymmetric; "S" = symmetric; "N" = all costs are equal to zero
b: Number of iterations
c: Sony VAIO notebook computer (1.83 GHz Intel Core 2 Duo processor)

Table 1: Summary of the computational results

4 Conclusions

We have developed a new linear programming model of the TSP. The formulation is an order of size smaller than that of the existing model, and results in computational times that are several orders of magintude smaller. Hence, we believe the proposed formulation represents a move in the right direction with respect to eventually being able to solve more realistically-sized TSP's.

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